P-PRIMARY TORSION OF THE BRAUER GROUP

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— Study group outline —

Let K be a Henselian field of characteristic 0 with residue field k of characteristic $p > 0$. We do not want to assume that k is perfect. Let $\mathrm{Br}(K) = H^2(K, \overline{K}^\times)$ be the Brauer group. This is a torsion group, so we have

$$\mathrm{Br}(K) = \mathrm{Br}(K)\{p\} \oplus \bigoplus_{\ell \neq p} \mathrm{Br}(K)\{\ell\}\,.$$

The Kummer sequence gives that $\mathrm{Br}(K_{\mathrm{nr}})$ is a p-primary torsion group. Then one can define the residue map r that fits into an exact sequence

$$0 \to \mathrm{Br}(k)\{\ell\} \to \mathrm{Br}(K)\{\ell\} \xrightarrow{r} H^1(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0\,. \tag{1}$$

This sequence is split by the choice of a uniformiser. If k is perfect, then $\mathrm{Br}(K_{\mathrm{nr}}) = 0$, so there is a similar exact sequence for all of $\mathrm{Br}(K)$, but in general this is not so. One task of the study group is to understand Kato's description of $\mathrm{Br}(K)\{p\}$ by means of a filtration $\mathrm{fil}_\bullet \mathrm{Br}(K)\{p\}$ in terms of the Swan conductor (defined using symbols), see [Kat89]. Exercise: relate a similar filtration on $H^1(K, \mathbb{Q}/\mathbb{Z})$ to the upper ramification filtration of $\mathrm{Gal}(\overline{K}/K)$. Note that $\mathrm{fil}_0\mathrm{Br}(K)\{p\}$ is exactly the subgroup of elements that die in $\mathrm{Br}(K_{\mathrm{nr}})$, so $\mathrm{fil}_0\mathrm{Br}(K)\{p\}$ fits into an exact sequence like (1).

To be able to work with the Brauer group of a variety we need a more general version of the above, where fields are replaced by rings. Understanding the filtration on the Brauer group of a ring requires understanding the p-adic vanishing cycles spectral sequence [BK86] (which reduces questions about the generic fibre to the special fibre, i.e., char 0 to char p) and the de Rham–Witt complex [Ill79]. All this leads to the definition of the refined Swan conductor rsw_n that allows one to understand the graded factors $\mathrm{fil}_n\mathrm{Br}(K)/\mathrm{fil}_{n+1}\mathrm{Br}(K)$.

Now let F be a p-adic field and let X be a smooth, projective variety over F with good reduction. Let $\mathcal{X} \to \mathrm{Spec}\,\mathcal{O}_F$ be a smooth and proper scheme with generic fibre X. Let $K = F(X)$. The filtration $\mathrm{fil}_n\mathrm{Br}(K^{\mathrm{h}})$, where h stands for henselisation, pulls back to a filtration on $\mathrm{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)$. The piece $\mathrm{fil}_0\mathrm{Br}(X)$ behaves like the prime-to-p part: one has the residue map $r : \mathrm{fil}_0\mathrm{Br}(X) \to H^1_{\text{ét}}(\mathcal{X}_0, \mathbb{Q}/\mathbb{Z})$, where $\mathcal{X}_0$ is the special fibre. This allows one to compute the evaluation of elements of $\mathrm{fil}_0\mathrm{Br}(X)$ on F-points of X by specialising the residue at the reduction of the point. The main achievement of [BN23] is the calculation of evaluation at F-points of X of the Brauer elements that belong to higher pieces $\mathrm{fil}_n\mathrm{Br}(X)$, $n \geq 1$, in terms of the refined Swan conductor, see [BN23, Theorem B]. The proof uses the analysis of behaviour of the refined Swan conductor under blowing-up.

This has nice applications to the Brauer–Manin obstruction:

For a variety X over a field k, an element $\mathcal{A} \in \mathrm{Br}(X)$ and a k-algebra R we define the *evaluation map* $\mathrm{ev}_{\mathcal{A}} : X(R) \to \mathrm{Br}(R)$ by $\mathrm{ev}_{\mathcal{A}}(P) = \mathcal{A}(P)$.

Let k be a number field, let Ω_k be the set of all places of k, and let $\mathbb{A}_k$ be the ring of adèles of k. For a subset $S \subset \Omega_k$ we denote by $\mathbb{A}_k^S$ the adèles without components for the places in S. By local class field theory we have the local invariant $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$, which is an isomorphism for non-archimedean v, injective onto $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ if $k_v \simeq \mathbb{R}$, and zero if $k_v \simeq \mathbb{C}$. The local invariant allows us to think of $\mathrm{ev}_{\mathcal{A}}$ as a map $X(k_v) \to \mathbb{Q}/\mathbb{Z}$. The standard spreading-out argument [CTS21, Proposition 13.3.1] based on the fact that $\mathrm{Br}(\mathcal{O}_F) = 0$, where $\mathcal{O}_F$ is the ring of integers of a p-adic field F, shows that the map

$$\mathrm{ev}_{\mathcal{A}} : X(\mathbb{A}_k) \to \prod_{v\in\Omega_k} \mathbb{Q}/\mathbb{Z} \tag{2}$$

factors through the direct $\bigoplus v \in \Omega_k \mathbb{Q}/\mathbb{Z}$. Thus we have a well defined pairing, called the *Brauer-Manin pairing*,

$$X(\mathbb{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$$

sending $(M_v)_{v\in\Omega_k}$ and $\mathcal{A} \in \mathrm{Br}(X)$ to the sum $\sum_{v\in\Omega_k} \mathrm{ev}_{\mathcal{A}}(M_v) \in \mathbb{Q}/\mathbb{Z}$ (which is actually a finite sum). The *Brauer-Manin set* $X(\mathbb{A}_k)^{\mathrm{Br}}$ is the left kernel of (2). We can also consider larger sets $X(\mathbb{A}_k)^B$, where $B \subset \mathrm{Br}(X)$.

It is natural to ask: which places of k show up in the Brauer–Manin set?

Definition 0.1. A place v of k is irrelevant if $\mathrm{ev}_{\mathcal{A}} : X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$ is constant for all $\mathcal{A} \in \mathrm{Br}(X)$.

Lemma 0.2. *Assume $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$. Let S be a set of places of k containing at least one non-archimedean place. The following conditions are equivalent:*

(i) All primes not in S are irrelevant.
(ii) The set $X(\mathbb{A}_k^S)$ is a direct factor of $X(\mathbb{A}_k)^{\mathrm{Br}}$.

If this holds, then $X(\mathbb{A}_k)^{\mathrm{Br}} = Z \times X(\mathbb{A}_k^S)$ where Z is a closed subset $X(\mathbb{A}_k^{\Omega_k\setminus S})$.

Recall that a variety Y over a perfect field of characteristic p is *ordinary* if $H^j(Y, B_Y^i) = 0$ for all i and j, where $B_Y^i := \mathrm{Im}[\Omega_Y^{i-1} \xrightarrow{d} \Omega_Y^i]$ is the sheaf of exact i-forms. For example, a K3 surface Y over a finite field $\mathbb{F}$ is *ordinary* if and only if the trace of Frobenius acting on $H^2_{\text{ét}}(Y_{\overline{\mathbb{F}}}, \mathbb{Q}_\ell)$, $\ell \neq p$, is not divisible by p.

The following is [BN23, Theorem C].

Theorem 0.3. (Bright-Newton) *Let X be a smooth, projective and geometrically integral variety over a number field k such that $H^2(X, \mathcal{O}_X) \neq 0$. Then every prime v of k of good, ordinary reduction, with residue characteristic p, it potentially relevant: there exists a finite extension K/k, a place w of K over v, and an element $\mathcal{A} \in \mathrm{Br}(X_K)\{p\}$ such that the evaluation map $\mathrm{ev}_{\mathcal{A}} ; X(K_w) \to \mathbb{Q}/\mathbb{Z}$ is non-constant.*

Sketch of proof: Let $i : \mathrm{Spec}\,\mathbb{F}_v \to \mathrm{Spec}\,\mathcal{O}_v$ and $\mathrm{Spec}\,k_v \to \mathrm{Spec}\,\mathcal{O}_v$ be the natural closed and open immersions, respectively. Let $\mathcal{X} \to \mathrm{Spec}\,\mathcal{O}_v$ be a smooth proper morphism with generic fibre X_v and special fibre $X_{\mathbb{F}_v}$. By an abuse of notation, we denote by $\bar{i}$ and $\bar{j}$ the embeddings of $X_{\overline{\mathbb{F}}_v}$ and of $\overline{X}_v$, respectively, into the pullback of $\mathcal{X}$ to the ring of integers of $\bar{k}_v$.

Consider the spectral sequence of p-adic vanishing cycles

$$H^n_{\text{ét}}(X_{\overline{\mathbb{F}}_v}, \bar{i}^* R^m \bar{j}_* \mathbb{Z}/p^r(1)) \Rightarrow H^{n+m}_{\text{ét}}(\overline{X}_v, \mathbb{Z}/p^r(1)),$$

and similar sequences with coefficients in $\mathbb{Z}_p(1)$ and $\mathbb{Q}_p(1)$. Let $\mathrm{gr}_0 H^2_{\text{ét}}(\overline{X}_v, \mathbb{Q}_p(1))$ be the image of

$$H^2_{\text{ét}}(\overline{X}_v, \mathbb{Q}_p(1)) \to H^0_{\text{ét}}(X_{\overline{\mathbb{F}}_v}, \bar{i}^* R^2 \bar{j}_* \mathbb{Q}_p(1)).$$

Using the assumption that $X_{\overline{\mathbb{F}}_v}$ is ordinary, the Hodge-Tate decomposition [BK86, Theorem 0.7(iii)] gives an isomorphism of $\mathrm{Gal}(\overline{k}_v/k_v)$-modules

$$\mathrm{gr}_0 H^2_{\text{ét}}(\overline{X}_v, \mathbb{Q}_p(1)) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong H^0(X_v, \Omega^2) \otimes_{k_v} \mathbb{C}_p(-2),$$

where $\mathrm{Gal}(\overline{k}_v/k_v)$ naturally acts on $\mathbb{C}_p$, the completion of $\overline{k}_v$. Thus the assumption $H^2(X, \mathcal{O}_X) \neq 0$ implies that $\mathrm{gr}_0 H^2_{\text{ét}}(\overline{X}_v, \mathbb{Q}_p(1)) \neq 0$. It follows that the image of

$$H^2_{\text{ét}}(\overline{X}_v, \mathbb{Z}/p^r(1)) \to H^0_{\text{ét}}(X_{\overline{\mathbb{F}}_v}, \bar{i}^* R^2 \bar{j}_* \mathbb{Z}/p^r(1))$$

is non-zero for some $r \geq 1$. Take an element of $H^2_{\text{ét}}(\overline{X}_v, \mathbb{Z}/p^r(1))$ with non-zero image. It comes from $H^2_{\text{ét}}(\overline{X}, \mathbb{Z}/p^r(1))$, because the natural map between these groups is an isomorphism by proper base change. After a finite extension of k we may assume that it comes from $H^2_{\text{ét}}(X, \mathbb{Z}/p^r(1))$, thus giving a desired Brauer class.

Let $K = k_v(X)$ and let K^{h} be the henselisation of K for the discrete valuation inherited from k_v. We have a spectral sequence of vanishing cycles

$$H^n_{\text{ét}}(X_{\mathbb{F}_v}, i^* R^m j_* \mathbb{Z}/p^r(1)) \Rightarrow H^{n+m}_{\text{ét}}(X_v, \mathbb{Z}/p^r(1)).$$

A similar sequence in Galois cohomology is

$$H^n(\mathbb{F}_v(X_{\mathbb{F}_v}), H^m(K^{\mathrm{h}}_{\mathrm{nr}}, \mathbb{Z}/p^r(1)) \Rightarrow H^{n+m}(K^{\mathrm{h}}, \mathbb{Z}/p^r(1)).$$

These sequences are compatible under restriction to the generic point, so we get a commutative diagram

$$\begin{array}{ccc} H^2_{\text{ét}}(X_v, \mathbb{Z}/p^r(1)) & \longrightarrow & H^0_{\text{ét}}(X_{\mathbb{F}_v}, i^* R^2 j_* \mathbb{Z}/p^r(1)) \\ \downarrow & & \downarrow \\ H^2(K^{\mathrm{h}}, \mathbb{Z}/p^r(1)) & \longrightarrow & H^2(K^{\mathrm{h}}_{\mathrm{nr}}, \mathbb{Z}/p^r(1)) \end{array}$$

One proves that the right-hand vertical map is injective [BN23, Lemma 3.4]. (This is non-trivial. The case $r = 1$ is due to Bloch–Kato [BK86, Proposition 6.1(i)] which was originally proved using Gabber's injectivity results for étale cohomology generalising work of Bloch-Ogus on the Gersten conjecture.) So our Brauer class gives an element of $H^2(K^{\mathrm{h}}, \mathbb{Z}/p^r(1))$ with non-zero image in $H^2(K^{\mathrm{h}}_{\mathrm{nr}}, \mathbb{Z}/p^r(1))$, so it's not in fil_0 of Kato's filtration (This is due to Kato, see [BN23, Proposition 2.6]. Recall that fil_0 is the source of the residue map. Note that in the ℓ-adic situation fil_0 is the whole group, for $\ell \neq p$.) Using the crucial relation between the refined Swan conductor and evaluation map [BN23, Theorem B], Bright and Newton show in [BN23, Theorem A] that the Brauer elements that have constant evaluation maps over all extensions of k_v, give rise to elements of $\mathrm{fil}_0 H^2(K^{\mathrm{h}}, \mathbb{Z}/p^r(1))$. □

By being slightly more precise, the same argument can be used to show that in this result one can take $\mathcal{A} \in \mathrm{Br}(X_K)[p]$, see [Pag23, Theorem 4.5].

For a prime v of k we denote by p_v the residue characteristic of k_v and by e_v the absolute ramification index of k_v.

Theorem 0.4. (Bright-Newton) *Let X be a smooth, projective and geometrically integral variety over a number field k such that $\mathrm{Pic}(X_{\overline{k}})$ is torsion-free. If v is a prime of good reduction for X such that $e_v < p_v - 1$ and $H^0(X_{\mathbb{F}_v}, \Omega^1) = 0$, then v is irrelevant.*

Proof. This is [BN23, Theorem D]. One shows that in these assumptions we have $\mathrm{Br}(X) = \mathrm{fil}_0\mathrm{Br}(X)$. For this one needs to show that the refined Swan conductor is zero on $\mathrm{fil}_n\mathrm{Br}(X)$ where $n \geq 1$. This is deduced from explicit formulae describing the action of multiplication by p on Kato's filtration fil_n and on the refined Swan conductor in terms of the Cartier operator on differential forms, see [BN23, Section 2] and [Pag23, Section 3]. For $\mathcal{A} \in \mathrm{Br}(X)\{\ell\}$, $\ell \neq p$, the statement follows from the fact that $X_{\overline{\mathbb{F}}_v}$ has no connected unramified cyclic covering of degree ℓ. □

In particular, for a K3 surface over $\mathbb{Q}$, odd primes of good reduction are irrelevant, see [BN23, Remark 7.5]. M. Pagano showed that for K3 surfaces the prime 2 can be relevant. In the ordinary case there is a somewhat stronger version:

Theorem 0.5. (M. Pagano) *Let X be a smooth, projective and geometrically integral variety over a number field k. Let v be a prime of k at which X has good ordinary reduction. Assume that $H^0(X_{\mathbb{F}_v}, \Omega^1) = 0$ and $H^1(X_{\overline{\mathbb{F}}_v}, \mathbb{Z}/p_v) = 0$. If $p_v - 1$ does not divide e_v, then v is irrelevant.*

There is a result applicable to non-ordinary reduction of K3 surfaces:

Theorem 0.6. (M. Pagano, Theorem 1.4) *Let X be a K3 surface with good non-ordinary reduction at v. If $e_v \leq p_v - 1$, then v is irrelevant.*

— Schedule —

14:00-15:30, Week 1: Room 342, Week 2-11: Room 140

1. **Overview + organisational meeting** (4 October) Alexei and Oli

2. **The logarithmic Hodge-Witt sheaf** (11 October) Alex

Let k be a perfect field of characteristic $p > 0$ and let X be a smooth scheme over k. As predicted by Milne, the logarithmic Hodge-Witt sheaf $W_r\Omega^q_{X,\log}$ plays the role of p-adic counterpart to the ℓ-adic sheaf $\mu_{\ell^r}^{\otimes q}$. It is defined to be the subsheaf of the Deligne-Illusie de Rham-Witt sheaf $W_r\Omega^q_X$ which is étale locally generated by sections $d\log[x_1]_r \wedge \cdots \wedge d\log[x_q]_r$ for $x_1, \ldots, x_q \in \mathcal{O}_X^*$. For example, the $d\log$ map factors through the Milnor K-theory sheaf $\mathcal{K}^{\mathrm{Mil}}_{q,X}$ and the p-adic analogue of the norm residue theorem is the Bloch-Gabber-Kato theorem [BK86]:

$$d\log\, : \mathcal{K}^{\mathrm{Mil}}_{q,X}/p^r \xrightarrow{\sim} W_r\Omega^q_{X,\log}\,.$$

The aim of this talk is to first define the de Rham-Witt complex $W_r\Omega^\bullet_X$ with the maps d, F, V, R following [Ill79, I. 1]. Then recall the canonical isomorphism

$$\mathbb{H}^*(X, W_r\Omega^\bullet_X) \cong H^*_{\mathrm{cris}}(X/W_r(k))\,.$$

[Ill79, II.1 Théorème 1.4]. Finally, define the subsheaf $W_r\Omega^q_{X,\log}$ and show that it is the kernel of the surjective map

$$1 - F : W_r\Omega^q_X \twoheadrightarrow W_r\Omega^q_X / dV^{r-1}\Omega^q_X$$

(see [CTSS83, Lemme 2].)

3. **p-torsion in the Brauer group** (18 October) Ambrus

Before moving to the mixed characteristic situation, we will first convince ourselves of the relevance of p-adic cohomology to the study of Brauer groups by studying the case where X is defined over a field k of the characteristic $p > 0$. In this setting, at least in the smooth case, the philosophy is that invariants of X should have to do with the slopes of Frobenius on the crystalline cohomology of X. For example, suppose that we wish to study the p^r-torsion of $\mathrm{Br}(X)$. The reason that this is more difficult (and more interesting) than for prime-to-p-torsion is that the Kummer sequence for the multiplication-by-p^r map fails to be exact on the right in the the étale topology. But it *is* exact for the fppf topology, and hence $\mathrm{Br}(X)[p^r]$ is a quotient of $H^2(X_{\mathrm{fl}}, \mu_{p^r})$, with kernel $\mathrm{NS}(X)/p^r$. Flat cohomology groups are difficult to work with, but for smooth X the natural map

$$R^1\epsilon_*\mu_{p^r} \to W_r\Omega^1_{X,\log}$$

is an isomorphism for all $r \geq 1$, where $\epsilon : X_{\mathrm{fl}} \to X_{\text{ét}}$ denotes the restriction of topoi. In particular, $H^2(X_{\mathrm{fl}}, \mu_{p^r}) \cong H^1(X, W_r\Omega^1_{X,\log})$, and since $W_r\Omega^1_{X,\log}[-1]$ sits inside the de Rham-Witt complex $W_r\Omega^\bullet_X$, taking hypercohomology gives a map $H^1(X, W_r\Omega^1_{X,\log}) \to H^2_{\mathrm{cris}}(X/W_r(k))$. This gives the link between $\mathrm{Br}(X)[p^r]$ and $H^2_{\mathrm{cris}}(X/W_r(k))$. The task of this talk is to present this in more detail following [Ill79, II.5]. In general things can be quite complicated, but one might like to present the theory through the classic example of a K3 surface X of finite height h, where you find that

$$\mathrm{Br}(X_{\overline{k}})[p^\infty] \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{22-\rho-2h}$$

where ρ is the geometric Picard number of X.

4. **p-adic vanishing cycles - local results** (25 October) Yuan

Now we move to the mixed characteristic situation relevant to [BN23]. Let K be a complete discrete valuation field of characteristic 0 with valuation ring $\mathcal{O}_K$ and (not necessarily perfect) residue field k of characteristic $p > 0$. Let X be a smooth and proper scheme over $\mathcal{O}_K$. The goal of this talk and the next is to understand the filtration on the p-adic étale cohomology $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$ (as Galois representations) coming from the p-adic vanishing cycles spectral sequence, at least in the case when the special fibre X_k is ordinary. This is the topic of [BK86]. The aim of this talk is to understand the local structure of the vanishing cycles sheaf $i^*R^qj_*\mathbb{Z}/p^r(q)$ via the symbol filtration. These results will be used in Talk 5 and in Talk 6. The speaker should cover [BK86, Theorem 1.4], and give some idea of the proof (which is given in §2-§6, loc. cit.). Note that since the local rings on X_k are direct limits of smooth $\mathbb{F}_p$-algebras, the theory of $W_r\Omega^q_{X,\log}$ developed in Talk 2 carries over to the non-perfect situation.

5. **p-adic vanishing cycles - global results** (1 November) Nina

   Introduce the notion of an ordinary variety following [BK86, §7]. Then cover the results in §9, loc. cit. We will use these results in Talk 11 but note that we only need them in the easier case that the special fibre is ordinary, so you may wish to rely on the easier Corollary 8.2 rather than Theorem 8.1.

6. **The refined Swan conductor** (8 November) Oli
   For a ring R, let $\mathbb{Z}/n(q)$ be the usual q-th Tate twist of the constant sheaf $\mathbb{Z}/n$ on $R_{\text{ét}}$ if $n \in R^*$. If R is smooth over a field of characteristic $p > 0$ then write $n = p^r m$ with $(m, p) = 1$ and define

$$\mathbb{Z}/n(q) := \mathbb{Z}/m(q) \oplus W_r\Omega^q_{R,\log}[-q] \in D^b(R_{\text{ét}})\,.$$

   Define $H^q(R) := \varinjlim_n H^q(R_{\text{ét}}, \mathbb{Z}/n(q-1))$ whenever $n \in R^*$ or R is smooth over a field of characteristic $p > 0$.
   The aim of this talk is to present the material in [BN23, §2.1-§2.3], which recalls and generalises [Kat89]. Let K be a Henselian discrete valuation field with residue field F of characteristic $p > 0$. One defines an increasing filtration $\mathrm{fil}_\bullet H^q(K)$ and the Swan conductor of $\chi \in H^q(K)$ is defined to be the least $n \geq 0$ such that $\chi \in \mathrm{fil}_n H^q(K)$. The graded pieces of the filtration inject (via the *refined Swan conductor*) into $\Omega^q_F \oplus \Omega^{q-1}_F$. The main case of interest for applications to Brauer groups is the case $q = 2$ since then $H^2(K) = \mathrm{Br}(K)$ by the Kummer sequence, but the general case is also very interesting. If X is a variety over a field k, let K^h be the fraction field of the Henselisation of $\mathcal{O}_{X,k(X)}$. Then one defines a filtration $\mathrm{fil}_\bullet\mathrm{Br}(X)$ by taking the pre-image of $\mathrm{fil}_\bullet\mathrm{Br}(K^h)$ by the natural map.

7. **The residue map and the tame part** (15 November) Netan
   The first task in this talk is to define the residue map $\partial : \mathrm{fil}_0 H^q(K) \to H^{q-1}(F)$ and compare it to the classical residue map on Brauer groups in the case $q = 2$, following [BN23, §2.4-§2.5]. Next, let k be a finite extension of $\mathbb{Q}_p$ and let X be a smooth, geometrically irreducible variety over k with smooth model $\mathcal{X}$ over $\mathcal{O}_k$ and geometrically irreducible special fibre Y. Given $\mathcal{A} \in \mathrm{Br}(X)$ one has an evaluation map $\mathrm{ev}_{\mathcal{A}} : \mathcal{X}(\mathcal{O}_k) \to \mathrm{Br}(k)$. Following [BN23, §3], prove Proposition 3.1 and its corollaries which describe the evaluation map on the smallest part of Kato's filtration $\mathrm{fil}_0\mathrm{Br}(X)$.

8. **The refined Swan conductor and blowing-up** (22 November) Justin
   This talk will follow [BN23, §4-§5] where they study how the refined Swan conductor behaves under blowing-up the model $\mathcal{X}$ in a smooth point. This will be essential in Talk 10.

9. **Calculations for $\mathbb{P}^n$** (29 November) Peter
   This talk follows [BN23, §6] to describe the graded pieces of Kato's Swan conductor filtration on $H^1(U, \mathbb{Q}/\mathbb{Z}) := \varinjlim_n H^1(U, \mathbb{Z}/n)$ where $U = \mathbb{P}^n_k \backslash Z$ is the complement of a hyperplane Z inside projective space over a field k of

characteristic p. Along with Talk 8, this computation will be the other essential ingredient in Talk 10.

10. **Proof of Theorem B** (6 December) Alexei
The main achievement of [BN23] is Theorem B, which facilitates the computation of the evaluation map associated to elements in $\mathrm{Br}(X)$ of p-power order. (Note that the evaluation map for elements of order coprime to p is well-understood. See [CTS96], [CTS13] and [Bri15]). In fact, Bright-Newton prove the stronger Theorem 8.1. This talk will state and prove Theorem 8.1 using the calculations from Talk 8 and Talk 9. The proof is based on an inductive argument using a chain of blow-ups in which the Swan conductor decreases at the exceptional divisors (the projective spaces appearing in Talk 8) until reaching fil_0, whereupon the evaluation map is understood via Talk 7.

11. **Proof of Theorem A and applications to the Brauer-Manin obstruction** (13 December) Rachel
Now that we have Theorem B available for evaluating the evaluation map for p-power order elements of $\mathrm{Br}(X)$, one can ask for applications. For example, let k be a finite extension of $\mathbb{Q}_p$ and let X be a smooth, geometrically irreducible variety over k with smooth model $\mathcal{X}$ over $\mathcal{O}_k$ and geometrically integral special fibre Y. Given $\mathcal{A} \in \mathrm{Br}(X)$ one may ask when $\mathrm{ev}_{\mathcal{A}} : \mathcal{X}(\mathcal{O}_k) \to \mathrm{Br}(k)$ factors through $\mathcal{X}(\mathcal{O}_k) \to \mathcal{X}(\mathcal{O}_k/\pi^i)$ for any $i \geq 1$, where π is a uniformiser of $\mathcal{O}_k$. (Note that if $\mathcal{A}$ has order coprime to p then $\mathrm{ev}_{\mathcal{A}}$ factors through the special fibre [Bri15, §5]). With this in mind, define the *evaluation filtration* $\mathrm{Ev}_\bullet\mathrm{Br}(X)$ as follows:
For a finite extension k'/k of ramification index $e(k'/k)$ and uniformiser π', for $P \in \mathcal{X}(\mathcal{O}_{k'})$ and $r \geq 1$ let $B(P,r) \subset \mathcal{X}(\mathcal{O}_{k'})$ be the set of points the same reduction as P modulo π'^r. Define

$$\mathrm{Ev}_{-2}\mathrm{Br}(X) := \{\mathcal{A} \in \mathrm{Br}(X) \,|\, \forall k'/k, \mathrm{ev}_{\mathcal{A}} \text{ is zero on } \mathcal{X}(\mathcal{O}_{k'})\}$$

$$\mathrm{Ev}_{-1}\mathrm{Br}(X) := \{\mathcal{A} \in \mathrm{Br}(X) \,|\, \forall k'/k, \mathrm{ev}_{\mathcal{A}} \text{ is constant on } \mathcal{X}(\mathcal{O}_{k'})\}$$

and

$$\begin{aligned}\mathrm{Ev}_n\mathrm{Br}(X) := \{\mathcal{A} \in \mathrm{Br}(X) \,|\, &\forall k'/k \text{ finite}, \forall P \in \mathcal{X}(\mathcal{O}_{k'}),\\ &\mathrm{ev}_{\mathcal{A}} \text{ is constant on } B(P, e(k'/k)(n+1))\}\end{aligned}$$

for $n \geq 0$.

The first part of this talk will follow [BN23, §9] in proving their Theorem A, which describes $\mathrm{Ev}_\bullet\mathrm{Br}(X)$ in terms of Kato's filtration $\mathrm{fil}_\bullet\mathrm{Br}(X)$ and the refined Swan conductor. The proof uses Theorem B. Note that $\mathrm{Ev}_{-2}\mathrm{Br}(X)$ is the image of $\mathrm{Br}(\mathcal{X})$ inside $\mathrm{Br}(X)$ by [BN23, Corollary 3.7] (actually this is true more generally for $\mathcal{X}$ just regular and proper over $\mathcal{O}_k$ by the main result of [SS14]).

The second part will present implications in the study of the Brauer-Manin obstruction. One should state and prove Theorem C and Theorem D of [BN23], as described in the study group outline, following §11. Note that the proofs also use the results of Talk 5. If time permits, one might also discuss Theorem 1.3 and Theorem 1.4 of [Pag23].

References

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